2020A

Fall 2020/21

Lecture 4

- Meaning of Double Integral
- Polar coordinates.

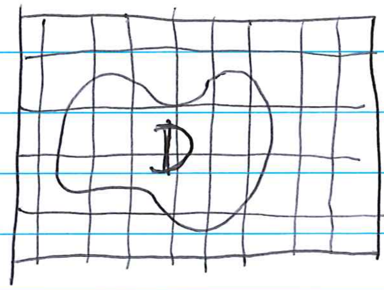
Let D be a region bdd by one or several piecewise smooth, closed simple curves. We define its area to be

$$\text{Area} \overset{\text{def}}{=} \iint_{R_0} \chi_D \quad ,$$

where $D \subset R_0$ rectangle and χ_D is the characteristic function of D.

We explained already that χ_D is integrable (because it is conti. except along the boundary curves of D).

Justification of definition



Let P be a partition of R_0. Divide the subrectgls of P into 3 classes:

$\mathcal{A}_1$: those lie inside the interior of D

$\mathcal{A}_2$: those touching the boundary of D

$\mathcal{A}_3$: those lie outside D.

Then, as $\chi_D = 0$ on $\mathcal{A}_3$,

$$S(\chi_D, \mathring{P}) = \sum_{\mathcal{A}_1} \chi_D \Delta x_j \Delta y_k + \sum_{\mathcal{A}_2} \chi_D \Delta x_j \Delta y_k$$

$$= \sum_{\mathcal{A}_1} \Delta x_j \Delta y_k + \sum_{\mathcal{A}_2} \chi_D(p_{jk}) \Delta x_j \Delta y_k$$

$\chi_D(P_{jk}) = 0$ or 1 depending whether $P_{jk} \notin D$ or $\in D$. In any case,

$$0 \leq \sum \chi_D(P_{jk}) \Delta x_j \Delta y_k$$

$$\leq \text{Const.} \times \text{Length of boundary curves} \times \|P\|$$

$$\to 0 \quad \text{as} \quad \|P\| \to 0$$

So, we see that when $\|P\| \to 0$

$$\iint_D 1 \quad \text{or} \quad \iint_{R_0} \chi_D \quad \text{is very closed to} \quad \sum_{A_1} \Delta x_j \Delta y_k,$$

the approximate area of D. So it is reasonable to take

$$\iint_D 1$$

as the area of D.

Next, when $f \geq 0$,

$$\iint_D f$$

is the volume of the region $\{(x,y,z) : 0 \leq z \leq f(x,y), (x,y) \in D\}$

as clear from its Riemann sum.

When $f \geq 0$ is the density of some material occupying D

$\iint_D f$ is the total mass of the shape D.

Finally, for any integrable function f,

$$\frac{1}{|D|} \iint_D f$$

is the average of f over D. From

$$m \leq f(x,y) \leq M, \qquad m = \min_D f, \quad M = \max_D f$$

we deduce

$$m \leq \frac{1}{|D|} \iint_D f \leq M.$$

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Polar coordinates v.s. Cartesian coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r \geq 0,$$

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\frac{y}{x},$$

These formulas set up a map Φ between

$$\{(r,\theta) : r \geq 0, \theta \in \mathbb{R}\} \longrightarrow \{(x,y) : (x,y) \in \mathbb{R}^2\}$$

Φ is onto but not 1-1. Here

$$\Phi((0,\theta)) = (0,0),$$ that is, the vertical axis is mapped to a single pt $(0,0)$, and.

$$\Phi((r,\theta+2n\pi)) = \Phi((r,\theta)), \quad \forall n \in \mathbb{Z}$$

Usually we restrict Φ to a strip of width 2π to get "almost" 1-1. For example, let

$$S = \{(r,\theta) : r \geq 0, \ \theta \in [0, 2\pi]\}$$

Then Φ maps S onto $\mathbb{R}^2$, and 1-1 on its interior.

Using the Polar-Cartesian relation, any function $f = f(x,y)$ can be converted as a fcn $\hat{f} = \hat{f}(r,\theta)$ and vice versa.

$f(x,y)$	$\hat{f}(r,\theta)$
$x^2 + y\cos x$	$r^2\cos^2\theta + r\sin\theta\cos(r\cos\theta)$
$\sqrt{x^2+y^2}\, e^{y}$	$r e^{r\sin\theta}$

Below are some well-known curves described in these 2 coord.

- the circle of radius a centered at the origin

$x^2+y^2=a^2$ $\qquad\qquad$ $r=a$

- the circle of radius a centered at $(\frac{a}{2},0)$

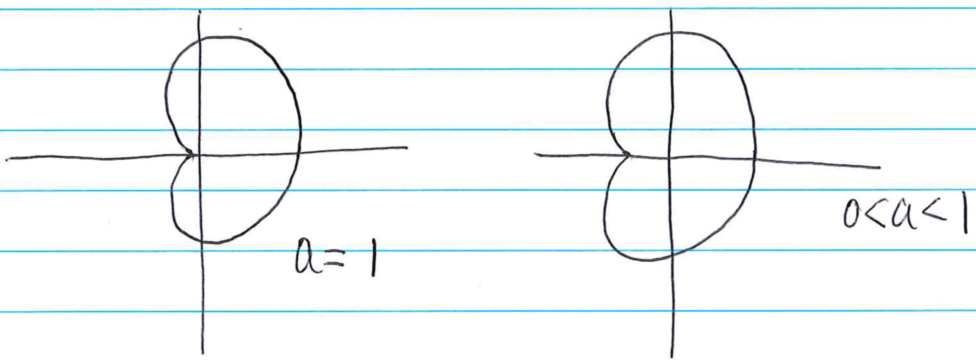
$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4} \qquad\qquad r=a\cos\theta$$

(why? $r=a\cos\theta \iff r^2=ar\cos\theta \iff x^2+y^2=ax \iff x^2-ax+y^2=0$

$\iff \left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$.)

- $cx=y,\ x\geq 0$ $\qquad\qquad$ $\theta=c$

- Cardioid $\quad r=1+a\cos\theta \quad (0<a\leq 1)$



$r^2=r+ar\cos\theta,\quad x^2+y^2=\sqrt{x^2+y^2}+ax$

$(x^2+y^2-ax)^2=x^2+y^2$ v. complicated in Cartesian coor.